数理解析研究所講究録
第 1655 巻 2009 年 124-131

A generalization of Weierstrass semigroups on a double covering of a curve [1]

神奈川工科大学・基礎・教養教育センター 米田 二良 (Jiryo Komeda)
Center for Basic Education and Integrated Learning
Kanagawa Institute of Technology

Abstract

Let $\pi : \tilde{C} \longrightarrow C$ be a double covering of a non-singular curve with a ramification point $\tilde{P}$. Let $H(\tilde{P})$ and $H(\pi(\tilde{P}))$ be the Weierstrass semigroups of the points $\tilde{P}$ and $\pi(\tilde{P})$ respectively. We extend the notions of $H(\tilde{P})$ and $H(\pi(\tilde{P}))$ to the numerical semigroups $\tilde{H}$ and H respectively, and classify the pairs of $(\tilde{H}, H)$ by their genera. Moreover, we study about the property of such a pair $(\tilde{H}, H)$ which means whether H (respectively $\tilde{H}$) is Weierstrass or not.

1 The d_2-map

Let $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ be the additive semigroup of non-negative integers. A subsemigroup H of $\mathbb{N}_0$ is called a *numerical semigroup* if its complement $\mathbb{N}_0 \backslash H$ in $\mathbb{N}_0$ is a finite set. The cardinality $\sharp(\mathbb{N}_0 \backslash H)$ is called the *genus* of H, which is denoted by $g(H)$. The symbols H and $\tilde{H}$ mean numerical semigroups throughout this paper. For any elements $a_1, \dots, a_m$ of $\mathbb{N}_0$ we denote by $\langle a_1, \dots, a_m \rangle$ the semigroup generatd by $a_1, \dots, a_m$. Let $\mathcal{H}$ be the set of numerical semigroups. We define the map $d_2 : \mathcal{H} \longrightarrow \mathcal{H}$ sending $\tilde{H}$ to $d_2(\tilde{H}) = \left\{ \frac{\tilde{h}}{2} \;\middle|\; \tilde{h} \in \tilde{H} \text{ is even} \right\}$, which is called *the d_2-map*.

Example 1. 1 i) $d_2 : \mathbb{N}_0 \longmapsto \mathbb{N}_0$. ii) $d_2 : \langle 2, 3 \rangle \longmapsto \mathbb{N}_0$.

iii) $d_2 : \langle 3, 4, 5 \rangle \longmapsto \langle 2, 3 \rangle$. iv) $d_2 : \langle 3, 5 \rangle \longmapsto \langle 3, 4, 5 \rangle$.

v) $d_2 : \langle 4, 6, 7 \rangle \longmapsto \langle 2, 3 \rangle$. vi) $d_2 : \langle 5, 7, 9 \rangle \longmapsto \langle 5, 6, 7, 8, 9 \rangle$.

vii) $d_2 : \langle 6, 8, 10, 11 \rangle \longmapsto \langle 3, 4, 5 \rangle$.

[1] This paper is an extended abstract and the details will appear elsewhere.

2 A geometric meaning of the d_2-map

A complete non-singular 1-dimensional algebraic variety over an algebraically closed field is abbreviated to a *curve* in this paper. Let (C, P) be a pointed curve and $k(C)$ the field of rational functions on C. We define the *Weierstrass semigroup of* P as follows:

$$H(P) = \{n \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = nP\}.$$

A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C, P) such that $H = H(P)$.

Lemma 2.1 *Let $\pi : \tilde{C} \longrightarrow C$ be a double covering of a curve, i.e., the degree of $k(\tilde{C}) \supset k(C)$ is two, with a ramification point $\tilde{P}$. Then $d_2(H(\tilde{P})) = H(\pi(\tilde{P}))$.* (For example see Lemma 2 in [4])

A numerical semigroup $\tilde{H}$ is called *the double covering type*, abbreviated to *DC* if there exists a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ such that $\tilde{H} = H(\tilde{P})$.

Example 2.1 Let $\pi : \tilde{C} \longrightarrow \mathbb{P}^1$ be a double covering of the projective line $\mathbb{P}^1$. If $\tilde{P}$ is a ramification point of π, then $H(\tilde{P}) = \langle 2, 2g+1 \rangle$ where g is the genus of $\tilde{C}$. Hence, $\langle 2, 2g+1 \rangle$ is DC.

By the definition of DC we have the following:

Remark 2.2 *If $\tilde{H}$ is DC, then $\tilde{H}$ and $d_2(\tilde{H})$ are Weierstrass.*

Using Riemann-Hurwitz' formula we see the following:

Lemma 2.3 *If $\tilde{H}$ is DC, then $g(\tilde{H}) \geqq 2g(d_2(\tilde{H}))$.*

The following is the known fact which is due to Torres [8].

Remark 2.4 *If $\tilde{H}$ is a Weierstrass semigroup with $g(\tilde{H}) \geqq 6g(d_2(\tilde{H})) + 4$, then it is DC.*

Example 2.2 Let $\tilde{H} = \langle 6, 8, 33 \rangle$. Then $d_2(\tilde{H}) = \langle 3, 4 \rangle$. We have

$$g(\tilde{H}) = 22 \geqq 6 * 3 + 4 = 6g(\langle 3, 4 \rangle) + 4.$$

Hence, $\tilde{H}$ is DC, because it is Weierstrass.

A numerical semigroup $\tilde{H}$ is said to be *lower-Weierstrass*, abbreviated to *ℓ-Weierstrass* if $d_2(\tilde{H})$ is Weierstrass. The definition of DC means the following:

Remark 2.5 *If $\tilde{H}$ is DC, then it is ℓ-Weierstrass.*

Remark 2.6 $B = \langle 13, 14, 15, 16, 17, 18, 20, 22, 23\rangle$ *is non-Weierstrass* (see [1]), *but ℓ-Weierstrass, because $d_2(B) = \langle 7, 8, 9, 10, 11, 13\rangle$ is of genus 7, which implies that $d_2(B)$ is Weierstrass* (see [3]).

3 Classification and existence

By Lemma 2.3 and Remark 2.4 we have the following table:

Table I : Numerical semigroups $\tilde{H}$

Genus	*Weierstrass*			*Non-Weierstrass*	
$6g+4 \leqq \tilde{g}$	xii) *DC*	$\not\exists$ *non-DC, ℓ-Wei*	$\not\exists$ *non-ℓ-W*	vi) *ℓ-Wei*	iii) *non-ℓ-W*
$2g \leqq \tilde{g} \leqq 6g+3$	xi) *DC*	x) *non-DC, ℓ-Wei*	viii) *non-ℓ-W*	v) *ℓ-Wei*	ii) *non-ℓ-W*
$\tilde{g} \leqq 2g-1$	$\not\exists$ *DC*	ix) *non-DC, ℓ-Wei*	vii) *non-ℓ-W*	iv) *ℓ-Wei*	i) *non-ℓ-W*

Here we set $\tilde{g} = g(\tilde{H})$ and $g = g(d_2(\tilde{H}))$.

We note that the bigger the roman numeral numbering the boxes in the table, the more special a numerical semigroup $\tilde{H}$ belonging to the box numbered by it. After deleting the boxes in Table I to which no numerical semigroup belongs, the above table becomes the following:

Table II : Numerical semigroups $\tilde{H}$

Genus	*Weierstrass*			*Non-Weierstrass*	
$6g+4 \leqq \tilde{g}$	xii) *DC*			vi) *ℓ-Wei*	iii) *non-ℓ-W*
$2g \leqq \tilde{g} \leqq 6g+3$	xi) *DC*	x) *non-DC, ℓ-Wei*	viii) *non-ℓ-W*	v) *ℓ-Wei*	ii) *non-ℓ-W*
$\tilde{g} \leqq 2g-1$	ix) *non-DC, ℓ-Wei*		vii) *non-ℓ-W*	iv) *ℓ-Wei*	i) *non-ℓ-W*

We have the following problem:

Problem A. Is a Weierstrass semigroup $\tilde{H}$ ℓ-Weierstrass ? Namely, is there no numerical semigroup belonging to the box numbered by viii) (respectively vii) ?

Problem B. Is there a Weierstrass semigroup which belongs to the box numbered by x) ?

Problem C. Is there a non-Weierstrass semigroup which belongs to the box numbered by vi) ?

We will show that some numerical semigroup belongs to each box except vi), vii), viii) and x).

3.1 Special Cases

The following is known:

Remark 3.1 ([7]) *Let H be a Weierstrass semigroup and n an odd number $\geqq 4g(H) - 1$. We set $\tilde{H} = 2H + n\mathbb{N}_0$. Then $d_2(\tilde{H}) = H$ and $\tilde{H}$ is DC. In this case we have $g(\tilde{H}) = 2g(H) + \dfrac{n-1}{2} \geqq 4g(H) - 1$.*

Hence this remark shows the existence of a numerical semigroup belonging to the box numbered by xii) (resp. xi))

Remark 3.2 ([6]) *Let $\tilde{H} = \langle 2n, 2n + 2 \times 1 - 1, \ldots, 2n + 2 \times n - 1\rangle$ with $n \geqq 3$. Then $\tilde{H}$ is Weierstrass and $d_2(\tilde{H}) = \langle n, 2n+1, \ldots, 2n+n-1\rangle$, which is Weierstrass. Hence, $\tilde{H}$ is ℓ-Weierstrass. In this case we have $g(\tilde{H}) = \dfrac{3}{2}g(H) + 1 \leqq 2g(H) - 1$.*

The numerical semigroups in Remark 3.2 are in the box numbered by ix). Let $a, b \in \mathbb{N}_0$ with $a < b$. The symbol $a \longrightarrow b$ stands for consecutive numbers $a, a+1, \ldots, b$. We know the following result:

Remark 3.3 ([5]) *Let $\tilde{H}_g = \langle 2g - 1 \longrightarrow 4g - 10, 4g - 8, 4g - 6, 4g - 5\rangle$ for $g \geqq 7$. Then it is non-Weierstrass.*

It is not difficult to show the following:

Proposition 3.4 *Let $\tilde{H}_g$ be as in Remark 3.3. Then $d_2(\tilde{H}_g) = \langle g \longrightarrow 2g - 3, 2g-1\rangle$, which is Weierstrass. In this case we have $g(\tilde{H}_g) = 2g(d_2(\tilde{H}_g))+2$. $\tilde{H}_7$ is the numerical semigroup in Remark 2.6.*

Hence this proposition shows that the box numbered by v) contains the above numerical semigroups.

3.2 General Cases

By Remark 2.4 we see the following:

Proposition 3.5 *Let H be a non-Weierstrass semigroup and n an odd number $\geqq 8g(H)+9$. We set $\tilde{H} = 2H + n\mathbb{N}_0$. Then $\tilde{H}$ is non-Weierstrass. In this case we have $g(\tilde{H}) = 2g(H) + \dfrac{n-1}{2} \geqq 6g(H)+4$.*

Thus, the above numerical semigroups belong to the box numbered by iii). A numerical semigroup H is said to be *primitive* if the largest integer in $\mathbb{N}_0 \backslash H$ is less than twice the least positive integer in H.

Example 3. 1 The numerical semigroup $H = \langle 13 \longrightarrow 18, 20, 22, 23 \rangle$ is primitive, because $\mathbb{N}_0 \backslash H = \{1 \longrightarrow 12, 19, 21, 24, 25\}$.

Example 3. 2 The numerical semigroup $H = \langle 13, 15 \longrightarrow 18, 20, 22, 23 \rangle$ is non-primitive, because $\mathbb{N}_0 \backslash H = \{1 \longrightarrow 12, 14, 19, 21, 24, 25, 27\}$.

We call H an *n-semigroup* if n is the least positive integer in H.

Lemma 3.6 *Let H be a primitive n-semigroup. We set*

$$\mathbb{N}_0 \backslash H = \{1 \longrightarrow n-1, l_n < l_{n+1} < \cdots < l_{g(H)}\}.$$

Take odd integers $\gamma_{n+1} < \gamma_{n+2} < \cdots < \gamma_{n+m}$ between $2n$ and $4n$. Let $\tilde{H}$ be a subset of $\mathbb{N}_0$ such that

$$\mathbb{N}_0 \backslash \tilde{H} = \{2, 4, \ldots, 2(n-1), 2l_n, 2l_{n+1}, \ldots, 2l_{g(H)}\}$$
$$\cup \{1, 3, \ldots, 2n-1, \gamma_{n+1}, \gamma_{n+2}, \ldots, \gamma_{n+m}\}$$

Then $\tilde{H}$ is a primitive $2n$-semigroup of genus $g(H)+n+m$ with $d_2(\tilde{H}) = H$.

For a numerical semigroup H we set $L_2(H) = \{l + l' \mid l, l' \in \mathbb{N}_0 \backslash H\}$. The following remark is well-known:

Remark 3.7 ([1]) *A numerical semigroup H with $\sharp L_2(H) \geqq 3g(H) - 2$ is non-Weierstrass.*

Example 3. 3 In Lemma 3.6 let $H = \langle 13 \longrightarrow 18, 20, 22, 23 \rangle$, $m = 1$ and $\gamma_{14} = 51$. In this case, $\tilde{H}$ is a primitive 26-semigroup such that

$$\mathbb{N}_0 \backslash \tilde{H} = \{1 \longrightarrow 25\} \cup \{38, 42, 48, 50\} \cup \{51\}.$$

Hence, $g(\tilde{H}) = 30 = 2g(H) - 2$. We have $\sharp L_2(\tilde{H}) = 88 = 3g(\tilde{H}) - 2$, which implies that $\tilde{H}$ is non-Weierstrass.

Hence this example belongs to the box numbered by i)

Example 3.4 In Lemma 3.6 let $H = \langle 13 \longrightarrow 18, 20, 22, 23 \rangle$, $m = 3$ and $\gamma_{14} = 43$, $\gamma_{15} = 49$, $\gamma_{16} = 51$. In this case, $\tilde{H}$ is a primitive 26-semigroup such that

$$\mathbb{N}_0 \backslash \tilde{H} = \{1 \longrightarrow 25\} \cup \{38, 42, 48, 50\} \cup \{43, 49, 51\}.$$

Hence, $g(\tilde{H}) = 32 = 2g(H)$. We have $\sharp L_2(\tilde{H}) = 94 = 3g(\tilde{H}) - 2$, which implies that $\tilde{H}$ is non-Weierstrass.

Thus, the box numbered by ii) contains the above numerical semigroup.

Lemma 3.8 ([2]) *Let H be a primitive numerical semigroup such that $\mathbb{N}_0 \backslash H = \{1 \longrightarrow 13, 15, 18, 27\}$, i.e., $H = \langle 14, 16, 17, 19 \longrightarrow 26, 29 \rangle$. Then H is Weierstrass.*

Example 3.5 *First Step.* In Lemma 3.6 let $H = \tilde{H}_0 = \langle 14, 16, 17, 19 \longrightarrow 26, 29 \rangle$, $m = 1$ and $\gamma_{n+1} = 55$. In this case, $\tilde{H}_1 = \tilde{H}$ is a primitive 28-semigroup such that

$$\mathbb{N}_0 \backslash \tilde{H} = \{1 \longrightarrow 27\} \cup \{30, 36, 54\} \cup \{55\}.$$

Hence, $g(\tilde{H}) = 31 = 2g(H) - 1$. We have $\sharp L_2(\tilde{H}) = 88 = 3g(\tilde{H}) - 5$.

Second Step. In Lemma 3.6 let $H = \tilde{H}_1$, $m = 1$ and $\gamma_{n+1} = 111$. In this case, $\tilde{H}_2 = \tilde{H}$ is a primitive 56-semigroup such that

$$\mathbb{N}_0 \backslash \tilde{H} = \{1 \longrightarrow 55\} \cup \{60, 72, 108, 110\} \cup \{111\}.$$

Hence, $g(\tilde{H}) = 60 = 2g(H) - 2$. We have $\sharp L_2(\tilde{H}) = 177 = 3g(\tilde{H}) - 3$.

Third Step. In Lemma 3.6 let $H = \tilde{H}_2$, $m = 1$ and $\gamma_{n+1} = 223$. In this case, $\tilde{H}_3 = \tilde{H}$ is a primitive 56-semigroup such that

$$\mathbb{N}_0 \backslash \tilde{H} = \{1 \longrightarrow 111\} \cup \{120, 144, 216, 220, 222\} \cup \{223\}.$$

Hence, $g(\tilde{H}) = 117 = 2g(H) - 3$. We have $\sharp L_2(\tilde{H}) = 351 = 3g(\tilde{H})$, which implies that $\tilde{H}_3 = \tilde{H}$ is non-Weierstrass.

By the above three steps we get a sequence

$$\tilde{H}_3 \xrightarrow{d_2} \tilde{H}_2 \xrightarrow{d_2} \tilde{H}_1 \xrightarrow{d_2} \tilde{H}_0$$

where $\tilde{H}_0$ is Weierstrass, $\tilde{H}_3$ is non-Weierstrass and $g(\tilde{H}_i) \leqq 2g(\tilde{H}_{i-1}) - 1$ for $i = 1, 2, 3$.

(1) If $\tilde{H}_1$ is non-Weierstrass, then it belongs to the box numbered by iv).

(2) If $\tilde{H}_1$ is Weierstrass and $\tilde{H}_2$ is non-Weierstrass, then $\tilde{H}_2$ belongs to the box numbered by iv).

(3) If $\tilde{H}_1$ and $\tilde{H}_2$ are Weierstrass, then $\tilde{H}_3$ belongs to the box numbered by iv).

Hence the above shows that the box numbered by iv) contains some numerical semigroup.

References

[1] R.O. Buchweitz, *On Zariski's criterion for equisingularity and non-smoothable monomial curves.* preprint 113, University of Hannover, 1980.

[2] J. Komeda, *On primitive Schubert indices of genus g and weight $g-1$*, J. Math. Soc. Japan **43** (1991) 437-445.

[3] J. Komeda, *On the existence of Weierstrass gap sequences on curves of genus* ≤ 8, J. Pure Appl. Alg. **97** (1994) 51-71.

[4] J. Komeda, *Cyclic coverings of an elliptic curve with two branch points and the gap sequences at the ramification points*, Acta Arithmetica **LXXXI** (1997) 275-297.

[5] J. Komeda, *Non-Weierstrass numerical semigroups*, Semigroup Forum **57** (1998) 157-185.

[6] J. Komeda, *Weierstrass semigroups whose minimum positive integers are even*, Arch. Math. **89** (2007) 52-59.

[7] J. Komeda and A. Ohbuchi, *On double coverings of a pointed non-singular curve with any Weierstrass semigroup,* Tsukuba J. Math. **31** (2007) 205–215.

[8] F. Torres, *Weierstrass points and double coverings of curves with application: Symmetric numerical semigroups which cannot be realized as Weierstrass semigroups,* Manuscripta Math. **83** (1994) 39-58.